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Sturm–Liouville solutions of the wave equation

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Abstract. The Sturm–Liouville solutions of the 3D-wave equation when the refractive index depends only on the radial variable r are deduced from solutions of the 1D-wave equation that the Laplace and Fourier transforms change into a Sturm–Liouville equation.

1. Introduction

We discuss here a particular class of solutions of the wave equation obtained in the following way from the solutions of a Sturm–Liouville equation.

Let us consider the wave equation

$$\partial_x^2\psi+\partial_y^2\psi+\partial_z^2\psi-n^2(x,y,z)\partial_{x_0}^2\psi=0 \qquad x_0=ct \tag{1}$$

where the refractive index n is some function of x, y, z. Then, generalizing Bateman's result for n constant [1] we obtain the Sturm–Liouville solutions of (1) when n depends on the radial variable $r=(x^2+y^2+z^2)^{1/2}$.

Let $F(z, x_0)$ be a solution of the 1D-wave equation

$$\partial_z^2F-n^2(z)\partial_{x_0}^2F=0. \tag{2}$$

Then, one checks easily that

$$\varphi(x,z,x_0)=(x\pm \mathrm{i}z)^{-1/2}F(\sqrt{x^2+z^2},x_0) \tag{3}$$

is a solution of the 2D-wave equation

$$\partial_x^2\varphi+\partial_z^2\varphi-n^2(x,z)\partial_{x_0}^2\varphi=0 \tag{4a}$$

in which the refractive index has the form

$$n(x,z)=n(x^2+z^2)^{1/2}. \tag{4b}$$

In a similar way if $\varphi(x, z, x_0)$ is a solution of ($4a$) for $n(x, z)$ arbitrary, then

$$\psi(x,y,z,x_0)=(x\pm iy)^{-1/2}\varphi((x^2+y^2)^{1/2},z,x_0) \tag{5a}$$

is a solution of (1) for the refractive index

$$n(x,y,z)=n((x^2+y^2)^{1/2},z). \tag{5b}$$

Then, combining (3) and (5) we obtain the following solutions of the wave equation (1)

$$\psi(x,y,z,x_0)=A(x,y,z)F(r,x_0) \tag{6}$$

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with

$$A(x, y, z) = (x \pm iy)^{-1/2}(\rho \pm iz)^{-1/2} \tag{7a}$$

where

$$\rho = (x^2 + y^2)^{1/2} \qquad r = (x^2 + y^2 + z^2)^{1/2} \tag{7b}$$

provided that the refractive index is

$$n(x, y, z) = n(r). \tag{8}$$

We call (6) the Sturm–Liouville solutions of the wave equation since the function F is solution of (2) that the Laplace and Fourier transforms change into a Sturm–Liouville equation.

A feature worth noticing in these solutions is the unusual attenuation factor (7a) $A(x, y, z)$. For instance, for n constant the solutions (6) are spherical waves, different from the usual ones, with the attenuation factor r^{-1} and reminiscent of the so-called electromagnetic missiles [2].

The Sturm–Liouville solutions in a cylindrical medium are given by the expression (3) with the attenuation factor $(x \pm iz)^{-1/2}$.

We now discuss the solutions of (2).

2. Time-arbitrary dependent solutions

To solve (2) we first use the Laplace transform [3]

$$F(z, s) = \int_0^\infty e^{-st} F(z, t)\, dt \tag{9}$$

with, to simplify, the same notation for the original function and its image.

Assuming $F(z, 0) = (\partial_t F(z, t))_{t=0} = 0$ the Laplace transform (9) changes the wave equation (2) into the Sturm–Liouville equation

$$\partial_z^2 F(z, s) - s^2 n^2(z) F(z, s) = 0 \tag{10}$$

with the general solution [4]

$$F(z, s) = \gamma^{-1/4}(z, s) \exp\left\{\int \gamma^{1/2}(z, s)\, dz\right\} \tag{11a}$$

where

$$\gamma(z, s) = -s^2 n^2(z) + c''(z, s) c^{-1}(z, s) \tag{11b}$$

while the function $c(z, s)$ satisfies the differential equation

$$c'' c^{-1} - s^2 n^2 = c^{-4}. \tag{12}$$

Neglecting $c''c^{-1}$ in (11b) gives the WKJB approximation. But for some expressions of $n(z)$ met in optics one may solve the Sturm–Liouville equation (9) exactly. Let us give two examples. For $n(z) = e^{-az}$ the solution of (10) is

$$F(z, s) = I_0\left(\frac{s}{a} e^{-az}\right) \tag{13}$$

where I_0 is the modified Bessel function of the first kind of order zero and the inverse Laplace transform of (13) is [5]

$$F(z, x_0)=\frac{1}{\pi}\left(\frac{2}{a}\,e^{-az}\left(x_0-\frac{e^{-az}}{a}\right)-\left(x_0-\frac{e^{-az}}{a}\right)^2\right)^{-1/2} H\left(\frac{2\,e^{-az}}{a}-x_0\right)H\left(x_0-\frac{e^{-az}}{a}\right) \tag{14}$$

where H is the Heaviside function.

Consequently, according to (6), when $n(r)=e^{-ar}$ equation (1) has the Sturm–Liouville solutions

$$\psi(x, y, z, x_0)=\frac{A(x, y, z)}{\pi}\left(\frac{2}{a}\,e^{-ar}\left(x_0-\frac{e^{-ar}}{a}\right)-\left(x_0-\frac{e^{-ar}}{a}\right)^2\right)^{-1/2}$$
$$\times H\left(\frac{2\,e^{-ar}}{a}-x_0\right)H\left(x_0-\frac{e^{-ar}}{a}\right). \tag{15}$$

As a second example we assume $n(z)=(1+az)^{-2}$. Then, the solution of (10) is

$$F(z, s)=(1+az)\exp\left(\frac{-s}{a(1+az)}\right) \tag{16}$$

with the inverse Laplace transform

$$F(z, t)=(1+az)\delta\left(x_0-\frac{1}{a(1+az)}\right) \tag{17}$$

where δ is the Dirac distribution. So, according to (6) when $n(r)=(1+ar)^{-2}$ equation (1) has the Sturm–Liouville solution

$$\psi(x, y, z, x_0)=A(x, y, z)(1+ar)\delta\left(x_0-\frac{1}{a(1+ar)}\right). \tag{18}$$

This Dirac pulse has some unusual properties being reminiscent of the so-called 'Big Crunch' of a contracting universe at $r=0$ and $x_0=a^{-1}$.

In these two simple examples the inverse Laplace transform had an analytical expression, but generally one has to compute numerically the inverse Laplace transform [6].

3. Time-harmonic solutions

For time-harmonic solutions of (2)

$$F(z, x_0)=e^{ikx_0}G(z) \tag{19}$$

Table 1. Time-harmonic solutions $F(z, x_0) = e^{ikx_0} G(z)$.

	$n(z)$	$G(z)$
1.	$(a+bz)^{-1}$:	$k^{-m}(a+bz)^m \qquad 2m-1=(1-4k^2/b^2)^{1/2}$
2.	$(a+bz+cz^2)^{-1}$:	$m^{-1/2}(a+bz+cz^2)^{1/2}\exp\left(\mathrm{i}m\int\frac{\mathrm{d}z}{a+bz+cz^2}\right) \qquad b^2-4ac=4(k^2-m^2)$
3.	$(a^2+z^2)^{-1}$:	$m^{-1/2}(a^2+z^2)^{1/2}\exp(\mathrm{i}m\tan^{-1}z/a) \qquad m^2=a^2\times k^2$
4.	e^{-az}:	$J_0\left(\frac{k}{a}e^{-az}\right)$
5.	$(C_0+C_1 e^{-2az})^{1/2}$:	$J_p\left(\frac{k}{a}C_1^{1/2}e^{-az}\right) \qquad p=\mathrm{i}\frac{k}{a}C_0^{1/2}$
6.	$(1-a^2z^2)^{1/2}$:	$e^{1/2kaz^2}I_z^{-2m}e^{-kaz^2} \qquad 4m=1-k/a$
7.	$(1-k^2z^2)^{1/2}$:	$e^{-k^2z^2/2}$
8.	$(1+\cos^2 az)^{1/2}$:	$Ce_m(az,q) \qquad Se_m(az,q) \qquad z^2q=m^2=2k^2/a^2 \qquad m$ integer
9.	$(1+(1+az)^2)^{-1}$:	$(1+(1+az)^2)^{1/2}\exp\left(\frac{\mathrm{i}m}{a}\tan^{-1}(1+az)\right) \qquad m=(a^2+k^2)^{1/2}$
10.	$(1+az)^{-2}$:	$(1+az)\exp(\mathrm{i}k/a(1+az))$
11.	$1+az$:	$(1+az)^{1/2}J_{-1/4}\left(\frac{k}{a}(1+az)^2\right)$
12.	$(1+az)^{-p}$:	$(1+az)^{1/2}J_{1/2(p-1)}\left(\frac{k}{a}\left(\frac{1+az}{1-p}\right)^{1-p}\right)$
13.	$(1-k^2z^2)^{-1}$: $kz<1$	$m^{-1/2}\left(\frac{a+bs+cs^2}{chks}\right)^{1/2}\exp\left(\mathrm{i}\tan^{-1}\left(\frac{2as+b}{2m}\right)\right)$ $mz=\tanh ks$

In lines 4, 5, 11, 12, J_n is the Bessel function of the first kind of order n. For the refractive index (8), equation (8) becomes the Mathieu equation with periodic solutions for m integer

$$\partial_z^2 F + (m^2 + 16q\cos 2z)F = 0$$

Here we have used Whittaker's notation [8].

one has just to change s into $-\mathrm{i}k$ in the Sturm–Liouville equation (10) which becomes

$$\partial_z^2 G(z) + k^2 n^2(z) G(z) = 0. \tag{20}$$

As previously stated this last equation may be solved exactly for some particular expressions of $n(z)$. We give in table 1 a list (non-exhaustive) of such solutions, many of which may be found in [4] and [7].

Then, using (6) and the results of table 1 we obtain, for instance, the time-harmonic Sturm–Liouville solutions for $n(r) = e^{-ar}$

$$\psi(x, y, z, x_0) = A(x, y, z) J_0\left(\frac{k}{a}e^{-ar}\right) e^{\mathrm{i}kx_0} \tag{21}$$

where J_0 is the Bessel function of the first kind of order zero. For $n(r) = (1+ar)^{-2}$ we have

$$\psi(x, y, z, x_0) = A(x, y, z)(1+ar)\frac{\mathrm{i}k}{e^{a(1+ar)}} e^{\mathrm{i}kx_0}. \tag{22}$$

Let us now consider a periodic medium with the refractive index

$$n(r) = (1+\cos^2 ar)^{1/2}.$$

Then (20) becomes a Mathieu equation with the solutions [8]

$$Ce_m(az, q) \qquad z^2 q = m^2 = 2k^2/a^2, \; m \text{ integer}$$

leading to the Sturm–Liouville solutions of (1)

$$\psi(x, y, z, x_0) = A(x, y, z) Ce_m(ar, q)\, e^{ikx_0}. \tag{23}$$

Some approximate Sturm–Liouville solutions of the wave equation with $n = n(r)$ may be obtained by applying the WKBJ approximation to (10) and (20).

4. Discussion

Since $n(x, y, z) = n(r)$ the Sturm–Liouville waves propagate in a medium with spherical symmetry. But although the wavefronts are spherical the amplitude is not constant on a wavefront, since the attenuation factor $A(x, y, z)$ has no spherical symmetry. Moreover as already noticed $|A(x, y, z)| < r^{-1}$ so that the attenuation of the Sturm–Liouville waves is smaller with distance than the attenuation of classical spherical waves. So the Sturm–Liouville waves make it possible to understand how diverging spherical waves can propagate in a non-uniform way. This result could be useful for instance to explain the 'Big Bang' theory of inhomogeneities in the universe.

Remark. For time-harmonic fields and radial $n(r)$ one may obtain solutions of the 3D-wave equation (1) in terms of generalized Bremmer series [9] provided, of course, that the series converge. The solutions discussed here are different since they are generated by applying the Bateman theorem twice to solutions of the 1D-Helmholtz equation (20) in which $n(z)$ has the same expression as $n(r)$.

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